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New Expression for Solution of wave equation

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Abstract: **In this paper, the alternative expression to the solution of wave equation can be obtained by using the trigonometric identities.**

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1. INTRODUCTION

The wave equation [1] is the considerable second-order linear partial differential equation for that can be used. As we have seen in physics, such as sound waves, light waves and water waves. It originates in fields like acoustics, electromagnetic, and fluid dynamics. The wave equation is a hyperbolic partial differential equation. The wave equation in one space dimension can be derived in a variety of different physical settings.

Most notably, it can be acquired for the case of a string that is vibrating in a two-dimensional plane, with each of its elements being pulled in opposite directions by the force of tension [3]. Shi and Wang [2] used Fourier series theory coupled with the techniques of real analysis inequalities and investigated the existence and uniqueness of periodic solutions for a class of neutral differential equations with delay. In this paper, we derive the solution of wave equation in using trigonometric identities. The Fourier series is employed to find a solution.

2. PRELIMINARIES

Consider the wave equation with the initial condition in our problem. The Fourier series and the trigonometric identities are used to solve the problem.

Definition 2.1. If f is Riemann integral over $[-L, L]$, then the Fourier series of f is the series

$$\frac{A_0}{2}+\sum_{n=1}^{\infty}\left[A_n\cos\left(\frac{n\pi x}{L}\right)+B_n\sin\left(\frac{n\pi x}{L}\right)\right], \qquad (1)$$

where $A_n=\frac{1}{L}\int_{-L}^{L}f(x)\cos(\frac{n\pi x}{L})dx, \quad n=0, \quad 1, \quad 2, \quad \ldots$ and

$$B_n=\frac{1}{L}\int_{-L}^{L}f(x)\sin(\frac{n\pi x}{L})dx, \quad n=0, \quad 1, \quad 2, \quad \ldots$$

Then A_n and B_n are called the Fourier Coefficients of f .

Consider the following one dimensional wave equation with BCs and ICs

PDE: $u_{tt}=c^2u_{xx}$

BC: $u(0,t)=u(L,t)=0$

IC: $u(x,0)=f(x)$, $u_t(x,0)=g(x)$

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Solution: Using separation of variables, Let $u(x,t) = \varphi(x)G(t)$

$u_{tt} = c^2 u_{xx}$ Implies $u(x,t) = \varphi(x)G''(t) = c^2\varphi(x)''G(t)$

$$\frac{\varphi(x)''}{\varphi(x)} = \frac{G''(t}{c^2 G(t)} = -\lambda \text{ (let)}$$

$$\varphi''(t) + \lambda\varphi(t) = 0 \text{ and } G''(t) + \lambda c^2 G(t) = 0$$

Recall for $\varphi''(t) + \lambda\varphi(t) = 0$, $\varphi(0) = \varphi(L) = 0$, $\lambda_n = \left(\frac{n\pi}{L}\right)^2, n = 1, \quad 2, \quad \ldots$ is Eigen value.

$\varphi_n(x) = \sin(\sqrt{\lambda_n}\, x), n = 1, \quad 2, \quad \ldots$ Eigen function.

For $G''(t) + \lambda c^2 G(t) = 0$

The general solution is $G_n(t) = A_n \cos(\sqrt{\lambda_n}\, ct) + B_n \sin(\sqrt{\lambda_n}\, ct), \quad n = 1, \quad 2, \quad \ldots$

Therefore, $u(x,t) = \sum_{n=1}^{\infty}\left[A_n \cos\left(\frac{n\pi ct}{L}\right) + B_n \sin\left(\frac{n\pi ct}{L}\right)\right]\sin(\frac{n\pi}{L}x)$ is the general solution.

From ICs, $\quad u(x,0) = \sum_{n=1}^{\infty} A_n \sin(\frac{n\pi x}{L}) = f(x)$

$$\sum_{n=1}^{\infty} A_n \sin(\frac{n\pi x}{L})\sin(\frac{m\pi x}{L}) = f(x)\sin(\frac{m\pi x}{L})$$

Implies $\sum_{n=1}^{\infty} A_n \int_0^L \sin(\frac{n\pi x}{L})\sin(\frac{m\pi x}{L})dx = \int_0^L f(x)\sin(\frac{m\pi x}{L})dx$

For $m = n$, $\quad A_n \int_0^L \sin^2(\frac{n\pi x}{L})dx = \int_0^L f(x)\sin(\frac{n\pi x}{L})dx$

$$\frac{A_n}{2}\int_0^L (1 - \cos^2(\frac{n\pi x}{L})dx = \int_0^L f(x)\sin(\frac{n\pi x}{L})dx$$

$$\frac{A_n}{2}\left[x - \frac{L}{2n\pi}\sin^2(\frac{n\pi x}{L})\right]_0^L = \int_0^L f(x)\sin(\frac{n\pi x}{L})dx$$

$$\frac{A_n}{2}(L) = \int_0^L f(x)\sin(\frac{n\pi x}{L})dx$$

$$A_n = \frac{2}{L}\int_0^L f(x)\sin(\frac{n\pi x}{L})dx$$

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Now, to find b_n since

$$u(x,t)=\sum_{n=1}^{\infty}\left[A_n\cos\left(\frac{n\pi ct}{L}\right)+B_n\sin\left(\frac{n\pi ct}{L}\right)\right]\sin(\frac{n\pi}{L}x)$$

$$u_t(x,t)=\sum_{n=1}^{\infty}\left[-A_n\frac{n\pi c}{L}\sin\left(\frac{n\pi ct}{L}\right)+B_n\frac{n\pi c}{L}\cos\left(\frac{n\pi ct}{L}\right)\right]\sin(\frac{n\pi}{L}x)$$

$$u_t(x,0)=\sum_{n=1}^{\infty}B_n\frac{n\pi c}{L}\sin(\frac{n\pi}{L}x)=g(x)$$

From ICs
$$\sum_{n=1}^{\infty}B_n\frac{n\pi c}{L}\sin(\frac{n\pi x}{L})\sin(\frac{m\pi x}{L})=g(x)\sin(\frac{m\pi x}{L})$$

$$\sum_{n=1}^{\infty}B_n\frac{n\pi c}{L}\int_0^L\sin(\frac{n\pi x}{L})\sin(\frac{m\pi x}{L})dx=\int_0^L g(x)\sin(\frac{m\pi x}{L})dx$$

For $m=n$,

$$B_n\frac{n\pi c}{L}\int_0^L\sin^2(\frac{n\pi x}{L})dx=\int_0^L g(x)\sin(\frac{n\pi x}{L})dx$$

$$\frac{B_n}{2}\frac{n\pi c}{L}\int_0^L(1-\cos(\frac{2n\pi x}{L}))dx=\int_0^L g(x)\sin(\frac{n\pi x}{L})dx$$

$$\frac{B_n}{2}\frac{n\pi c}{L}(x-\frac{L}{2n\pi}\sin(\frac{2n\pi x}{L}))_0^L=\int_0^L g(x)\sin(\frac{n\pi x}{L})dx$$

$$\frac{B_n n\pi c}{2}=\int_0^L g(x)\sin(\frac{n\pi x}{L})dx$$

$$B_n=\frac{2}{n\pi c}\int_0^L g(x)\sin(\frac{n\pi x}{L})dx$$

$$B_n=\frac{2}{\sqrt{\lambda_n}cL}\int_0^L g(x)\sin(\sqrt{\lambda_n}x)dx$$

3. MAIN RESULT

We next show that $u(x,y)=R(x-ct)+S(x+ct)$, where R and S are functions of one variable is the solution of $u_{tt}=c^2u_{xx}$ in $-\infty<x<\infty$ and $-\infty<t<\infty$ that can be derived by using the Fourier series subjected to the initial condition $u(x,0)=f(x)$ and $u_x(x,0)=g(x)$ by using trigonometric identities.

Proof: Since

$$u(x,t)=\sum_{n=1}^{\infty}\left[A_n\sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi ct}{L}\right)+B_n\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi ct}{L}\right)\right]$$

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From trigonometric identities, we have:

$$\sin ax \cos bx = \frac{1}{2}\left(\sin(a+b)x + \sin(a-b)x\right)$$

$$\sin ax \sin bx = \frac{1}{2}\left(\cos(a-b)x - \cos(a+b)x\right)$$

This implies;

$$A_n \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi ct}{L}\right) = \frac{1}{2}\left(\sin(x+ct)\frac{n\pi}{L} + \sin(x-ct)\frac{n\pi}{L}\right)$$

$$B_n \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi ct}{L}\right) = \frac{1}{2}\left(\cos(x-ct)\frac{n\pi}{L} - \cos(x+ct)\frac{n\pi}{L}\right)$$

$$= \frac{1}{2}\cos(x-ct)\frac{n\pi}{L} - \frac{1}{2}\cos(x+ct)\frac{n\pi}{L}$$

$$B_n \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi ct}{L}\right) = \frac{1}{2}\cos(x-ct)\frac{n\pi}{L} - \frac{1}{2}\cos(x+ct)\frac{n\pi}{L}$$

= (wave travelling to the right) – (wave travelling to the left)

$$u(x,t) = \sum_{n=1}^{\infty}\left[A_n \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi ct}{L}\right) + B_n \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi ct}{L}\right)\right]$$

$$= \sum_{n=1}^{\infty}\frac{A_n}{2}\left[\left(\sin\frac{n\pi}{L}(x-ct) + \sin\frac{n\pi}{L}(x+ct)\right)\right]$$

$$+ \sum_{n=1}^{\infty}\frac{B_n}{2}\left[\left(\cos\frac{n\pi}{L}(x-ct) - \cos\frac{n\pi}{L}(x+ct)\right)\right]$$

$$u(x,0) = \sum_{n=1}^{\infty}\frac{A_n}{2}\left[\left(\sin\frac{n\pi}{L}x + \sin\frac{n\pi}{L}x\right)\right] + \sum_{n=1}^{\infty}\frac{B_n}{2}\left[\left(\cos\frac{n\pi}{L}x - \cos\frac{n\pi}{L}x\right)\right]$$

$$\Rightarrow \quad f(x) = u_1(x,0) + u_2(x,0)$$

Now:-

$$u_1(x,t) = \sum_{n=1}^{\infty}\left(\frac{A_n}{2}\sin\frac{n\pi}{L}(x-ct) + \frac{B_n}{2}\cos\frac{n\pi}{L}(x-ct)\right)$$

$$u_2(x,t) = \sum_{n=1}^{\infty}\left(\frac{A_n}{2}\sin\frac{n\pi}{L}(x+ct) - \frac{B_n}{2}\cos\frac{n\pi}{L}(x+ct)\right)$$

Implies $u_1(x,0) = f_1(x) = R(x)$ and $u_x(x,0) = g(x)$

Since $f(x) = f_1(x) + f_2(x)$, and

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$$f_1(x-ct)=\sum_{n=1}^{\infty}\left(\frac{A_n}{2}\sin\frac{n\pi}{L}(x-ct)+\frac{B_n}{2}\cos\frac{n\pi}{L}(x-ct)\right) \text{ and}$$

$$f_2(x+ct)=\sum_{n=1}^{\infty}\left(\frac{A_n}{2}\sin\frac{n\pi}{L}(x+ct)-\frac{B_n}{2}\cos\frac{n\pi}{L}(x+ct)\right)$$

Thus, $u_1(x,t)= f_1(x-ct)=R(x-ct)$ ds $u_2(x,t)= f_2(x+ct)=S(x+ct)$

Since $u(x,t)=u_1(x,t)+u_2(x,t)$, we have $u(x,t)=R(x-ct)+S(x+ct)$.

This completes the proof.

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